# Proof by Contrapositive

# The contrapositive.

**Important:** The contrapositive of an implication $P \implies Q$ is $\sim Q \implies \sim P$.

- If $\sim Q$ is false (meaning $Q$ is true) the implication $\sim Q \implies \sim P$ is automatically true.
- So we assume $\sim Q$ is true – that is, that $Q$ is false – and try to conclude that $\sim P$ is true – meaning that $P$ is false.

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If $\underbrace{a|b \text{ and } b|c}_{P}$ then $\underbrace{a|c}_{Q}$

If $a \nmid c$ then $\neg(a|b \text{ and } b|c)$

If $a \nmid c$ then $a \nmid b$ or $b \nmid c$

$a, b, c$

If $a \nmid c$ and $a|b$ then $b \nmid c$

# Contrapositive.

**Proposition:** Suppose that $x \in \mathbb{Z}$. Suppose $x^2 - 4x + 3$ is even. Then $x$ is odd.

Suppose $\underbrace{x^2-4x+3 \text{ is even}}_{P}$. then $\underbrace{x \text{ is odd}}_{Q}$.

$\neg Q$ $x$ is even

$\neg P$ $x^2-4x+3$ is odd

If $x$ is even then $x^2-4x+3$ is odd.

If $x$ is even then $x = 2k$ for some $k \in \mathbb{Z}$.

$$x^2-4x+3 = (2k)^2 - 4(2k) + 3$$

$$= 4k^2 - 8k + 3$$

$$= 2(4k^2 - 8k + 1) + 1$$

# Contrapositive

Since $x^2-4x+3 = 2h+1$ where $h = 4k^2-8k+1$, we see that $x$ is odd.

**Proof:** Suppose $x$ is even. Then $x = 2m$ for some integer $m$. Therefore

$$B = x^2 - 4x + 3 = 4m^2 - 8x + 3 = 2(2m^2 - 2m + 1) + 1.$$

Since $B$ is of the form $2k + 1$ with $k = 2m^2 - 2m + 1$, we conclude that $B$ is odd. Therefore $B$ is not even. We have shown that if $x$ is not odd, then $B$ is not even, and therefore if $B$ is even, $x$ is odd.

# Contrapositive

**Proposition:** Suppose that $x \in \mathbb{Z}$, that $a$ is even, and that $b$ is odd. If $x^2 - ax + b$ is even, then $x$ is odd.

If $a$ even, $b$ odd and $x^2 - ax + b$ is even then $x$ is odd.

If $x$ is even then $x^2 - ax + b$ is odd

$a$ even $b$ odd

$\vdots$

same argument as before

# An example from calculus

**Theorem:** Let $f : [a, b] \to \mathbb{R}$ be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval $(a, b)$. If $f'(x) = 0$ for all $x \in [a, b]$, then $f$ is constant.

**Proof:**

- We will show that if $f$ is not constant, then there is an $x \in [a, b]$ with $f'(x) \neq 0$.
- Suppose that $f(x)$ is not constant. Then there are two (different) points $u$ and $v$ in $[a, b]$ such that $f(u) \neq f(v)$.

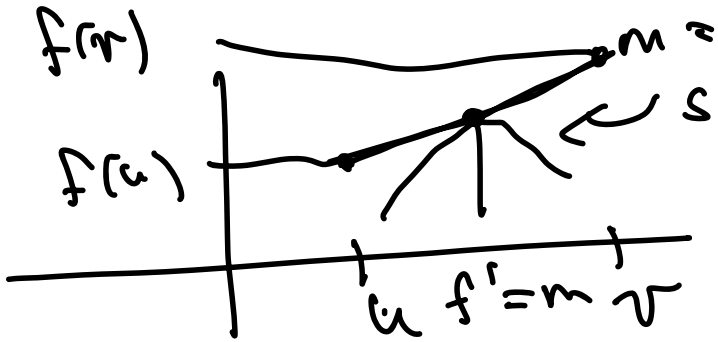
# calculus cont'd

- $f : [u, v] \to \mathbb{R}$ is continuous on $[u, v]$ and differentiable on $(u, v)$. Therefore, by the mean value theorem, there is a point $c \in (u, v)$ such that

$$f'(c) = \frac{f(v) - f(u)}{v - u}.$$

  Since $f(v) \neq f(u)$, the quantity on the right is not zero, and so $f'(c) \neq 0$.

- Therefore $f'(x)$ is not zero for all $x \in [a, b]$. This proves our result.




$m \neq 0$
so $f' \neq 0$
$\exists\, c \in (u, v)$ with $f' \neq 0$